LAPLACE TRANSFORM

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LAPLACE TRANSFORM

Let $f(t)$ be a function of t defined for all $t \geq 0$ .then the Laplace transform of $f(t)$, denoted by $L[\ f(t)]$ is defined by $L[\ f(t)] = \int_0^\infty e^{-st} f(t)dt$ Provided that the integral exists, "s" is a parameter which may be real or complex. Clearly $L[\ f(t)]$ is a function of s and is briefly written as $F(s)$ $(i.e.)$ $L[\ f(t)] = F(s)$

PIECEWISE CONTINUOUS FUNCTION

A function $f(t)$ is said to be piecewise continuous is an interval $a \leq t \leq b$, if the interval can be sub divided into a finite number of intervals in each of which the function is continuous and has finite right and left hand limits.

EXPONENTIAL ORDER

A function $f(t)$ is said to be exponential order if $\lim_{t\to\infty} e^{-st} f(t)$ is a finite quantity, where $s > 0$(exists).

PROPERTIES

- $L[af(t) \pm bg(t)] = aL[f(t)] \pm bL[g(t)]$, where a and b are constants.
- $If\ L[f(t)] = F(s), then\ L[f(at)] = \frac{1}{a} F(\frac{s}{a})\ ;\ a > 0$
- $If\ L[f(t)] = F(s), then$

  $i)\ L[e^{-at} f(t)] = F(s+a)$ and $ii)\ L[e^{at} f(t)] = F(s-a)$
- $L[f'(t)] = sL[f(t)] - f(0)$
- $L[f^n(t)] = s^n L[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - s^{n-3} f''(0) - \cdots f^{n-1}(0)$
- $IfL[f(t)] = F(s), then\ L\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}$
- $IfL\ [f(t)] = F(s), then$

  $i)\ L\ [tf(t)] = -\frac{d}{ds} F(s)$ and $ii)\ L\ [t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$

STANDARD FUNCTIONS

- $L(1) = \frac{1}{s}$
- $L(t) = \frac{1}{s^2}$
- $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$
- $L(e^{at}) = \frac{1}{s-a}, s > a$
- $L(e^{-at}) = \frac{1}{s+a}, s > a$
- $L(\sin at) = \frac{a}{s^2+a^2}, s > |a|$
- $L(\cos at) = \frac{s}{s^2+a^2}, s > |a|$
- $L(\sinh at) = \frac{a}{s^2-a^2}, s > |a|$
- $L(\cosh at) = \frac{s}{s^2-a^2}, s > |a|$

PERIODIC FUNCTIONS

Definition: A function $f(t)$ is said to be periodic if $(t + T) = f(t)$ for all values of t and for certain values of T. The smallest value of T for which $f(t + T) = f(t)$ for all t is called **periodic function**.

Example:

$sint = \sin(t + 2\pi) = \sin(t + 4\pi) \cdots$

$\therefore sint$ is periodic function with period 2π.

Let $f(t)$ be a periodic function with period T.

Then $L(f(t) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)$

Example:

Find the Laplace transform of $f(t) = \begin{cases} \sin \omega t, 0 < t < \frac{\pi}{\omega} \\ 0, \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}, f\left(t + \frac{2\pi}{\omega}\right) = f(t)$

Solution:

$$L(f(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\ dt = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} e^{-st}\, sin\, \omega t\ dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2}(-s \sin \omega t - \omega \cos \omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left\{\frac{e^{-s\frac{\pi}{\omega}}}{s^2+\omega^2}(-s \sin \pi - \omega \cos \pi) + \frac{\omega}{s^2+\omega^2}\right\} = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-s\frac{\pi}{\omega}}\omega + \omega}{s^2+\omega^2}\right]$$

$$= \frac{1}{1^2 - \left(e^{-\frac{\pi s}{\omega}}\right)^2} \left[\frac{\omega\left(e^{-s\frac{\pi}{\omega}} + 1\right)}{s^2 + \omega^2}\right] = \frac{1}{\left(1 + e^{-\frac{\pi s}{\omega}}\right)\left(1 - e^{-\frac{\pi s}{\omega}}\right)} \left[\frac{\omega\left(e^{-s\frac{\pi}{\omega}} + 1\right)}{s^2 + \omega^2}\right]$$

$$= \frac{\omega}{\left(1-e^{-\frac{\pi s}{\omega}}\right)(s^2+\omega^2)}$$

INVERSE LAPLACE TRANSFORM

Definition

If $L[f(t)] = F(s)$, then $f(t)$ is called an inverse Laplace transform of $F(s)$ and it is denoted by $f(t) = L^{-1}[F(s)]$, where $L^{-1}$is called the inverse Laplace transform operator.

$L(f(t))$	$F(s)$
$L(1) = \frac{1}{s}$	$L^{-1}\left(\frac{1}{s}\right) = 1$
$L(t) = \frac{1}{s^2}$	$L^{-1}\left(\frac{1}{s^2}\right) = t$
$L(t^n) = \frac{n!}{s^{n+1}}, if\ n\ is\ an\ integer$	$L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$ $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

$L(f(t))$	$F(s)$
$L(e^{at}) = \frac{1}{s-a}$	$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$
$L(e^{-at}) = \frac{1}{s+a}$	$L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$
$L(\sin at) = \frac{a}{s^2+a^2}$	$L^{-1}\left(\frac{a}{s^2+a^2}\right) = \sin at$
$L(\cos at) = \frac{s}{s^2+a^2}$	$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$
$L(\sinh at) = \frac{a}{s^2-a^2},$	$L^{-1}\left(\frac{a}{s^2-a^2}\right) = \sinh at$
$L(\cosh at) = \frac{s}{s^2-a^2}$	$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$

PROPERTIES

$L(f(t)) = F(s)$ and $L(g(t)) = G(s)$, then

1. $L^{-1}(aF(s) \pm bG(s)) = aL^{-1}(F(s)) \pm bL^{-1}(G(s))$, where a and b are constants.
2. $L^{-1}[F(s + a) = e^{at}L^{-1}[F(s)]]$
3. $L^{-1}[F(s - a) = e^{-at}L^{-1}[F(s)]]$
4. If $L^{-1}(F(s)) = f(t)$ and $f(0) = 0$ then $L^{-1}[sF(s)] = -\frac{d}{dt}L^{-1}F(s)$
5. $L^{-1}\left(\frac{F(s)}{s}\right) = \int_0^t L^{-1}(F(s))\, ds$
6. $L^{-1}(F(s)) = -\frac{1}{t}\ L^{-1}\left(\frac{d}{ds}F(s)\right)$
7. $L^{-1}(F(s)) = tL^{-1}\left(\int_s^\infty F(s)\, ds\right)$

Example:

Find $L^{-1}\left(\frac{1}{(s+1)(s+2)(s)}\right)$

Solution:

$$\left(\frac{1}{(s+1)(s+2)(s)}\right) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{c}{s}$$

$$1 = A(s+2)s + B(s+1)s + C(s+1)(s+2)$$

Put $s = -1, -2, 0$ then $A = -1$, $B = \frac{1}{2}$ and $C = \frac{1}{2}$

$$L^{-1}\left(\frac{1}{(s+1)(s+2)(s)}\right) = L^{-1}\left(\frac{-1}{s+1} + \frac{1/2}{s+2} + \frac{1/2}{s}\right)$$

$$= -L^{-1}\left(\frac{1}{s+1}\right) + 1/2\, L^{-1}\left(\frac{1}{s+2}\right) + 1/2\, L^{-1}\left(\frac{1}{s}\right)$$

$$= -e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2}$$


thank
you